Malaysian Journal of Mathematical Sciences 20(1): 423–445 (2026)
https://doi.org/10.47836/mjms.20.1.20

Malaysian Journal of Mathematical Sciences

Journal homepage: https://mjms.upm.edu.my

A Comprehensive Approach to Decision Making Problems Using Type-2 Pythagorean Fuzzy Sets

Paul, A.[1,2], John, S. J.[2], and Thankachan, B.*[3]

[1]*Department of Mathematics, Providence Women's College(Autonomous), Kozhikode, Kerala, India*
[2]*Department of Mathematics, National Institute of Technology Calicut, Kozhikode, Kerala, India*
[3]*Department of Mathematics, Manipal Institute of Technology,*
Manipal Academy of Higher Education, Manipal, Karnataka, India

E-mail: baiju.t@manipal.edu
**Corresponding author*

Received: 21 February 2025
Accepted: 28 July 2025

Abstract

Type-2 Pythagorean Fuzzy Sets (T2PFS) extend classical fuzzy frameworks by incorporating both primary and secondary membership measures, thereby capturing higher order uncertainty more effectively. This study presents a unified and interpretable algorithm for decision-making, supported by formally defined set-theoretic operations-union, intersection, and complement-for T2PFS. Additionally, novel score and hesitancy functions are proposed to enhance the ability to evaluate and prioritize alternatives. The algorithm is validated through a practical situation, and comparative analysis demonstrates improved precision and applicability over existing methods. These findings contribute to the development of more effective fuzzy decision-making models for uncertain environments.

Keywords: decision-making problems; evaluation function; hesitancy; Pythagorean fuzzy set; score function; type-2 Pythagorean fuzzy number; type-2 Pythagorean fuzzy set.

1 Introduction

Motivated by the advances and the expressive capacity of T2PFS, this paper aims to develop a unified and interpretable framework for decision-making under uncertainty. While prior studies have introduced various aggregation methods within the T2PFS environment, the fundamental set-theoretic operations-union, intersection, and complement-have not yet been explicitly defined, unlike in structures such as T2IFS. This omission hinders the theoretical maturity and broader application of T2PFS. Our paper addresses this foundational gap by formally defining these operations and proposing a robust decision-making algorithm. Additionally, existing score and accuracy functions often yield identical values for distinct alternatives, undermining ranking clarity. To overcome this, the study introduces new score and hesitancy functions with greater discriminative power, improving precision in uncertain decision-making environments.

The main objectives of this work are:

- To provide explicit and rigorous definitions for the union, intersection, and complement of T2PFS.
- To construct novel score and hesitancy functions capable of distinguishing between closely ranked alternatives.
- To develop a practical decision making algorithm using T2PFS.
- To illustrate the algorithm with an example and perform comparative analysis with existing methods.

To attain this objective, the study unfolds as follows: Section 2 presents the literature review, tracing the conceptual development from fuzzy sets through their various extensions to T2PFS. Section 3, presents the essential preliminary definitions that lay the foundation for a comprehensive understanding of the paper's content. Section 4 introduces the newly defined set-theoretic operations and discusses their properties. Section 5 develops the decision-making algorithm and illustrates its application using a real-world example. Section 6 provides a comparative analysis of the proposed method against existing techniques using score and hesitancy functions. The section also includes advantages of the proposed algorithm. Finally, Section 7 concludes the paper with a summary of our findings and suggestions for future research.

2 Related Works

In the realm of decision making and data analysis, traditional set theory initially served as the cornerstone for modeling and drawing conclusions. However, crisp distinctions soon proved inadequate for capturing the complexity and uncertainty inherent in real world scenarios. This limitation led to the introduction of fuzzy sets (FS) by Zadeh [27] in the 1960s, which revolutionized the modeling of vagueness by allowing gradual membership measures between 0 and 1. FS laid the groundwork for significant advancements in control systems, artificial intelligence, and decision sciences. For example, even in the recent years, Saidin et al. [20] introduced a divergence measure within a fuzzy TOPSIS framework to improve the precision of staff performance appraisals, demonstrating the adaptability of fuzzy models in real-world decision-making scenarios.

Earlier, as the field evolved, it became evident that some decision making circumstances required explicit modeling of exclusion and hesitation. In response, Intuitionistic Fuzzy Sets (IFS) were introduced by Atanassov [2] in 1983 and formally developed in 1986 [4]. IFS incorporated membership, non-membership, and a hesitancy margin, constrained such that their sum does not exceed 1. However, in many practical applications, decision-makers encountered situations where this sum exceeded 1. To address this, Yager [25] proposed Pythagorean Fuzzy Sets (PFS), where the square sum of membership and non-membership values must be less than or equal to 1, offering greater flexibility.

Meanwhile, Type-2 Fuzzy Sets (T2FS) [16] introduced a second-order uncertainty layer by making the membership value itself a fuzzy set. Alhough powerful, T2FS increased computational complexity. To simplify implementation, Mendel et al. [17] proposed Interval Valued Type-2 Fuzzy Sets (IVT2FS), where the primary membership is modeled as an interval and the secondary membership is fixed at 1. Around the same time, Atanassov [3] introduced Interval Valued Intuitionistic Fuzzy Sets (IVIFS), providing additional flexibility.

Extending this line of development, Peng and Yang [19] proposed Interval Valued Pythagorean Fuzzy Sets (IVPFS), introduced new aggregation operators and demonstrated their utility in multi-attribute decision making. More recently, a research paper by Al-Sabri et al. [1] presented a refined framework termed Pythagorean cubic fuzzy sets (PCFS), which built on IVPFS by representing membership and non-membership measures using cubic sets. In their study, novel aggregator operators derived from Einstein operators were introduced, offering a significant advancement in the solution of decision-making problems.

Further motivated by the growing importance of addressing both uncertainty and hesitancy, the concept of Type-2 Intuitionistic Fuzzy Sets (T2IFS) [6] was introduced. T2IFS are defined by four parameters-primary membership, primary non-membership, secondary membership, and secondary non-membership-providing finer granularity in complex decision-making problems. However, to overcome the limitations of T2IFS, researchers introduced Type-2 Pythagorean Fuzzy Sets (T2PFS) [18], which combine the flexible structure of PFS with the expressive depth of T2FS. T2PFS are characterized by six parameters: the primary membership, the primary non-membership, the sceondary membership and non-membership of the primary membership and the secondary membership and non-membership of the primary non-membership, enabling richer representation of ambiguity and preference.

Other advanced models have also emerged. For instance, the Fuzzy Parameterized Pythagorean Fuzzy Hypersoft Expert Set (FPPFHsES) [13] integrates fuzzy soft sets, hypersoft sets, and PFS while allowing criteria partitioning into subcriteria. While highly capable to create more adaptable expert systems, such models are computationally demanding and harder to interpret in practical applications.

Recent research has considerably expanded the applications of PFS, extending their use well beyond foundational contexts into more advanced domains. For instance, Wan Mohd et al. [23] applied PFS within an integrated AHP-VIKOR MCDM framework for green supplier development, demonstrating the practical effectiveness of PFS-based approaches in complex, sustainability-focused selection problems. Furthermore, a newer correlation coefficient was applied to disaster management and medical diagnosis, offering a fresh perspective in these critical fields [9]. Also, Ejegwa [10] introduced statistical techniques for computing correlation coefficients of PFSs, yielding promising results in medical diagnosis. Pythagorean fuzzy distance measures were developed and evaluated against existing approaches, demonstrating enhanced accuracy and effectiveness in pattern recognition tasks [24]. Similarly, Ejegwa [8] proposed a tri-parametric and weighted distance measure within the PFS framework, which outperformed conventional methods in decision-making applications, particularly in pattern classification and disease diagnosis.

More recently, Li et al. [15] proposed a similarity function to analyze player dynamics within Liverpool Football Club during the 2022/2023 EPL season, providing a comprehensive performance evaluation framework. Further advancing the practical scope of Pythagorean fuzzy models, Terna et al. [22] introduced a simple and intuitive Pythagorean fuzzy distance technique based on Euler's number. This method was successfully applied to multi-criteria decision-making in renewable energy source selection, highlighting its relevance in sustainable systems analysis. Additionally, a study by Yan et al. [26] enhanced the existing Pythagorean fuzzy correlation coefficient by developing a new Pythagorean fuzzy partial correlation coefficient (PFPCC), thereby introducing advanced tools for modeling relationships among uncertain variables. It was applied effectively in pattern recognition tasks to demonstrate the practical utility of the proposed measures. Hence, building on the advancements in FS theory, this study focuses on T2PFS and explores their potential applications in decision-making scenarios.

3 Preliminaries

This section delves into certain terminology and related notations crucial for the understanding of T2PFS. Throughout the paper X is the universe of discourse.

Definition 3.1. [7] *A type-2 fuzzy set (T2FS) is a fuzzy set whose membership measure corresponding to each element is also a fuzzy set. A T2FS $\tilde{A}$ can be defined as,*

$$\tilde{A} = \Big\{ << y, u >, \mu_{\tilde{A}}(< y, u >) >: \forall y \in X, \forall u \in J_y \subset [0, 1]) \Big\},$$

where $0 \leq \mu_{\tilde{A}} < y, u > \leq 1$ is the secondary membership of y with primary membership measure u and J_y is the domain of $\mu_{\tilde{A}} < y, u >$. A T2FS can be denoted as,

$$\tilde{A} = \bigsqcup_{y \in X} \bigsqcup_{u \in J_y} \mu_{\tilde{A}}(< y, u >)/_{<y,u>},$$

where $\bigsqcup_{y \in X} \bigsqcup_{u \in J_y}$ denotes union over all y and u.

Example 3.1. *Suppose a feedback of various aspects like mileage, comfort, performance, engine life and maintenance of a particular car is taken from three customers. Based on the feedback, a quantitative chart is made as follows. Let,*

$$\begin{aligned} &C_1 = \textit{mileage}, \qquad C_2 = \textit{comfort}, \qquad C_3 = \textit{performance}, \\ &C_4 = \textit{engine life}, \qquad C_5 = \textit{maintenance}, \qquad \textit{and} \\ &X = \{\textit{mileage, comfort, performance, engine life, maintenance}\}, \end{aligned}$$

the primary membership is the reliability of the customers on these aspects of the car and the secondary membership is the measure of reliability varying from whether it is very much reliable to not at all reliable. Suppose it is quantified as follows:

$$\begin{aligned} &J_{C_1} = \{0.70, 0.80, 0.90\}, \qquad J_{C_2} = \{0.70, 0.80, 0.60\}, \qquad J_{C_3} = \{0.30, 0.60, 0.40\}, \\ &J_{C_4} = \{1.00, 0.90, 0.80\}, \qquad J_{C_5} = \{0.30, 0.40, 0.45\}, \end{aligned}$$

and the secondary memberships are given by,

$$u_{C_1} = \left(\frac{0.70}{0.90} \sqcup \frac{0.80}{0.60} \sqcup \frac{0.90}{0.90}\right), \quad u_{C_2} = \left(\frac{0.70}{0.80} \sqcup \frac{0.80}{0.90} \sqcup \frac{0.60}{0.60}\right), \quad u_{C_3} = \left(\frac{0.30}{0.20} \sqcup \frac{0.60}{0.60} \sqcup \frac{0.40}{0.50}\right),$$
$$u_{C_4} = \left(\frac{1.00}{0.90} \sqcup \frac{0.90}{1.00} \sqcup \frac{0.80}{0.700}\right), \quad u_{C_5} = \left(\frac{0.30}{0.90} \sqcup \frac{0.40}{0.80} \sqcup \frac{0.45}{0.85}\right).$$

Then, the T2FS $\tilde{A}$ can be represented as,

$$\tilde{A} = \left(\frac{0.70}{0.90} \sqcup \frac{0.80}{0.60} \sqcup \frac{0.90}{0.90}\right) \Big/_{C_1} \sqcup \left(\frac{0.70}{0.80} \sqcup \frac{0.80}{0.90} \sqcup \frac{0.60}{0.60}\right) \Big/_{C_2} \sqcup \left(\frac{0.30}{0.20} \sqcup \frac{0.60}{0.60} \sqcup \frac{0.40}{0.50}\right) \Big/_{C_3}$$
$$\sqcup \left(\frac{1.00}{0.90} \sqcup \frac{0.90}{1.00} \sqcup \frac{0.80}{0.70}\right) \Big/_{C_4} \sqcup \left(\frac{0.30}{0.90} \sqcup \frac{0.40}{0.80} \sqcup \frac{0.45}{0.85}\right) \Big/_{C_5}.$$

Definition 3.2. [25] *A Pythagorean fuzzy set (PFS) is a fuzzy set whose square sum of the membership and non-membership measures is less than or equal to 1. A PFS $\tilde{P}$ can be defined as,*

$$\tilde{P} = \{< y, \alpha_{\tilde{P}(y)}, \beta_{\tilde{P}(y)} >: y \in X\},$$

where

$$\alpha_{\tilde{P}} : X \longrightarrow [0,1] \text{ is the membership measure of } y,$$
$$\beta_{\tilde{P}} : X \longrightarrow [0,1] \text{ is the non-membership measure of } y,$$

with $\alpha^2_{\tilde{P}}(y) + \beta^2_{\tilde{P}}(y) \leq 1, \forall y \in X$. A PFS can be denoted as,

$$\tilde{P} = \bigsqcup_{y \in X} < \alpha_{\tilde{P}(y)}, \beta_{\tilde{P}(y)} > \Big/_{y},$$

where $\bigsqcup_{y \in X}$ denotes union overall y.

Example 3.2. *Let $X = \{\tilde{y}, \tilde{z}\}$, $\alpha_{\tilde{P}(\tilde{y})} = 0.70$ and $\beta_{\tilde{P}(\tilde{y})} = 0.50$. Also let $\alpha_{\tilde{P}(\tilde{z})} = 0.80$ and $\beta_{\tilde{P}(\tilde{z})} = 0.40$, then the PFS is written as,*

$$\tilde{P} =< 0.70, 0.50 > \Big/_{\tilde{y}} \sqcup < 0.80, 0.40 > \Big/_{\tilde{z}}.$$

Definition 3.3. [18] *A type-2 Pythagorean fuzzy set $(T2PFS)$ is a fuzzy set that is type-2 and has also got primary membership and non-membership measures as well as secondary membership and non-membership measures both of whose square sum are less than or equal to 1.*

A T2PFS $\tilde{P}$ can be defined as,

$$\tilde{P} = \Big\{ < \Big(y, u(y), v(y)\Big), \Big(u(y), \alpha_{\tilde{P}}(u(y)), \beta_{\tilde{P}}(u(y))\Big), \Big(v(y), \alpha_{\tilde{P}}(v(y)), \beta_{\tilde{P}}(v(y))\Big) >$$
$$: y \in X, u \in {J_y}^m, v \in {J_y}^n \Big\}.$$

where ${J_y}^m$, ${J_y}^n \subseteq [0,1]$ are domains of the primary membership and non-membership measures of $y \in \tilde{P}$ respectively with $0 \leq u^2(y) + v^2(y) \leq 1$. Also the functions

$$\alpha_{\tilde{P}} : {J_y}^m, {J_y}^n \longrightarrow [0,1],$$
$$\beta_{\tilde{P}} : {J_y}^m, {J_y}^n \longrightarrow [0,1],$$

are secondary membership and non-membership measures respectively satisfying

$$0 \leq \alpha^2_{\tilde{P}}(u(y)) + \beta^2_{\tilde{P}}(u(y)) \leq 1 \;\forall\; u(y) \in {J_y}^m,$$
$$0 \leq \alpha^2_{\tilde{P}}(v(y)) + \beta^2_{\tilde{P}}(v(y)) \leq 1 \;\forall\; v(y) \in {J_y}^n.$$

Alternatively a T2PFS $\tilde{P}$ can be represented as,

$$\tilde{P} = \bigsqcup_{y \in X} \left(\bigsqcup_{u \in {J_y}^m, v \in {J_y}^n} \frac{< \alpha_{\tilde{P}}(u(y)), \beta_{\tilde{P}}(u(y)) >}{u(y)}, \frac{< \alpha_{\tilde{P}}(v(y)), \beta_{\tilde{P}}(v(y)) >}{v(y)} \right) \Big/_{y},$$

where $\bigsqcup_{y \in X} \; \bigsqcup_{u \in {J_y}^m, v \in {J_y}^n}$ *denotes the union over all $y \in X, u \in {J_y}^m, v \in {J_y}^n$.*

Example 3.3. *In the example of the T2FS above 3.1, let us include the primary and secondary non - memberships also to make it T2PFS. Then u_{C_1} in T2FS can be modified as,*

$$u_{C_1} = \frac{< 0.70, 0.00 >}{0.90}, \frac{< 0.80, 0.10 >}{0.20},$$

where 0.70 *and* 0.90 *were the original primary membership and secondary membership respectively. Here,*

$$u(y) = 0.90, \qquad v(y) = 0.20,$$
$$\alpha_{\tilde{P}}(u(y)) = 0.70, \qquad \beta_{\tilde{P}}(u(y)) = 0.00,$$
$$\alpha_{\tilde{P}}(v(y)) = 0.80, \qquad \beta_{\tilde{P}}(v(y)) = 0.10.$$

Similarly let,

$$u_{C_2} = \frac{< 0.70, 0.20 >}{0.80}, \frac{< 0.60, 0.50 >}{0.40}.$$

Then, the T2PFS of these 2 attributes can be represented as,

$$\tilde{P} = \left(\frac{< 0.70, 0.00 >}{0.90}, \frac{< 0.80, 0.10 >}{0.20} \right) \Big/_{C_1} \bigsqcup \left(\frac{< 0.70, 0.20 >}{0.80}, \frac{< 0.60, 0.50 >}{0.40} \right) \Big/_{C_2}.$$

Definition 3.4. *Let $\tilde{P}$ be a T2PFS. Then a Type-2 Pythagorean Number(T2PFN) is an element of the form,*

$$\tilde{p} = \left\{ < \Big(y, u(y), v(y) \Big), \Big(u(y), \alpha_{\tilde{P}}(u(y)), \beta_{\tilde{P}}(u(y)) \Big), \Big(v(y), \alpha_{\tilde{P}}(v(y)), \beta_{\tilde{P}}(v(y)) \Big) > \right\},$$

where $y \in X$, $0 \leq u^2(y) + v^2(y) \leq 1$,

$$0 \leq \alpha^2_{\tilde{P}}(u(y)) + \beta^2_{\tilde{P}}(u(y)) \leq 1 \;\forall\; u(y) \in {J_y}^m,$$
$$0 \leq \alpha^2_{\tilde{P}}(v(y)) + \beta^2_{\tilde{P}}(v(y)) \leq 1 \;\forall\; v(y) \in {J_y}^n.$$

Alternatively a T2PFN $\tilde{p}$ can be represented as,

$$\tilde{p} = \left(\frac{< \alpha_{\tilde{P}}(u(y)), \beta_{\tilde{P}}(u(y)) >}{u(y)}, \frac{< \alpha_{\tilde{P}}(v(y)), \beta_{\tilde{P}}(v(y)) >}{v(y)} \right) \Big/_{y}.$$

Example 3.4. *In the example of T2PFS above 3.3, the T2PFNs are*

$$\tilde{p_1} = \{< (C_1, 0.90, 0.20), (0.90, 0.70, 0.00), (0.20, 0.8, 0.10) >\}, \quad \textit{and}$$
$$\tilde{p_2} = \{< (C_2, 0.80, 0.40), (0.80, 0.70, 0.20), (0.40, 0.60, 0.50) >\},$$

or alternatively

$$\tilde{p_1} = \left(\frac{< 0.70, 0.00 >}{0.90}, \frac{< 0.80, 0.10 >}{0.20} \right) \Big/_{C_1}, \quad \textit{and}$$
$$\tilde{p_2} = \left(\frac{< 0.70, 0.20 >}{0.80}, \frac{< 0.60, 0.50 >}{0.40} \right) \Big/_{C_2}.$$

4 Some Properties of T2PFS

In this section, we formally define the core operations on T2PFS, namely union, intersection, and complement, and examine their fundamental properties. We further introduce two basic T2PFSs and establish a mathematical relationship between them. In addition, we define the score function and hesitancy for T2PFNs, and present a theorem that connects these concepts. To support the theoretical development, illustrative examples are provided for all definitions. Moreover, where certain properties do not hold under conventional T-norm and T-conorm operations, counterexamples are presented to demonstrate these limitations.

Let $\tilde{P}_1$ and $\tilde{P}_2$ be two T2PFS defined as,

$$\tilde{P}_1 = \Big\{ < \Big(y, u_1(y), v_1(y)\Big), \Big(u_1(y), \alpha_{\tilde{P}_1}(u_1(y)), \beta_{\tilde{P}_1}(u_1(y))\Big), \Big(v_1(y), \alpha_{\tilde{P}_1}(v_1(y)), \beta_{\tilde{P}_1}(v_1(y))\Big) > \\ : y \in X, u_1 \in {J_y}^m, v_1 \in {J_y}^n \Big\},$$

and

$$\tilde{P}_2 = \Big\{ < \Big(y, u_2(y), v_2(y)\Big), \Big(u_2(y), \alpha_{\tilde{P}_2}(u_2(y)), \beta_{\tilde{P}_2}(u_2(y))\Big), \Big(v_2(y), \alpha_{\tilde{P}_2}(v_2(y)), \beta_{\tilde{P}_2}(v_2(y))\Big) > \\ : y \in X, u_2 \in {J_y}^m, v_2 \in {J_y}^n \Big\}.$$

We define the union, intersection, and complement of T2PFS as follows;

Definition 4.1 (Union). *The T2PFS, $\tilde{P} = \tilde{P}_1 \bigcup \tilde{P}_2$ where,*

$$\tilde{P} = \Big\{ < \Big(y, u(y), v(y)\Big), \Big(u(y), \alpha_{\tilde{P}}(u(y)), \beta_{\tilde{P}}(u(y))\Big), \Big(v(y), \alpha_{\tilde{P}}(v(y)), \beta_{\tilde{P}}(v(y))\Big) > \\ : y \in X, u \in {J_y}^m, v \in {J_y}^n \Big\},$$

is defined as,

$$u(y) = \bigvee_{y \in X} \Big(u_1(y), u_2(y)\Big), \qquad v(y) = \bigwedge_{y \in X} \Big(v_1(y), v_2(y)\Big),$$
$$\alpha_{\tilde{P}}(u(y)) = \bigvee_{u(y)} \Big(\alpha_{\tilde{P}_1}(u_1(y)), \alpha_{\tilde{P}_2}(u_2(y))\Big), \qquad \beta_{\tilde{P}}(u(y)) = \bigwedge_{u(y)} \Big(\beta_{\tilde{P}_1}(u_1(y)), \beta_{\tilde{P}_2}(u_2(y))\Big),$$
$$\alpha_{\tilde{P}}(v(y)) = \bigvee_{v(y)} \Big(\alpha_{\tilde{P}_1}(v_1(y)), \alpha_{\tilde{P}_2}(v_2(y))\Big), \qquad \beta_{\tilde{P}}(v(y)) = \bigwedge_{v(y)} \Big(\beta_{\tilde{P}_1}(v_1(y)), \beta_{\tilde{P}_2}(v_2(y))\Big),$$

where $\bigvee_y$ denotes the fuzzy union over all y and $\bigwedge_y$ denotes the fuzzy intersection over all y.

Definition 4.2 (Intersection). *The T2PFS, $\tilde{P} = \tilde{P}_1 \bigcap \tilde{P}_2$ where,*

$$\tilde{P} = \Big\{ < \Big(y, u(y), v(y)\Big), \Big(u(y), \alpha_{\tilde{P}}(u(y)), \beta_{\tilde{P}}(u(y))\Big), \Big(v(y), \alpha_{\tilde{P}}(v(y)), \beta_{\tilde{P}}(v(y))\Big) > \\ : y \in X, u \in {J_y}^m, v \in {J_y}^n \Big\},$$

is defined as,

$$u(y) = \bigwedge_{y\in X}\Big(u_1(y), u_2(y)\Big), \qquad v(y) = \bigvee_{y\in X}\Big(v_1(y), v_2(y)\Big),$$

$$\alpha_{\tilde{P}}(u(y)) = \bigwedge_{u(y)}\Big(\alpha_{\tilde{P}_1}(u_1(y)), \alpha_{\tilde{P}_2}(u_2(y))\Big), \qquad \beta_{\tilde{P}}(u(y)) = \bigvee_{u(y)}\Big(\beta_{\tilde{P}_1}(u_1(y)), \beta_{\tilde{P}_2}(u_2(y))\Big),$$

$$\alpha_{\tilde{P}}(v(y)) = \bigwedge_{v(y)}\Big(\alpha_{\tilde{P}_1}(v_1(y)), \alpha_{\tilde{P}_2}(v_2(y))\Big), \qquad \beta_{\tilde{P}}(v(y) = \bigvee_{v(y)}\Big(\beta_{\tilde{P}_1}(v_1(y)), \beta_{\tilde{P}_2}(v_2(y))\Big),$$

where $\bigvee_y$ *denotes the fuzzy union over all* y *and* $\bigwedge_y$ *denotes the fuzzy intersection over all* y.

Definition 4.3 (Complement)**.** *The complement of the T2PFS,* $\tilde{P}$*, is given by,*

$$\tilde{P}^c = \Big\{ < \Big(y, v(y), u(y)\Big), \Big(v(y), \beta_{\tilde{P}}(v(y)), \alpha_{\tilde{P}}(v(y))\Big), \Big(u(y), \beta_{\tilde{P}}(u(y)), \alpha_{\tilde{P}}(u(y))\Big) > \\ : y \in X, v \in {J_y}^m, u \in {J_y}^n\Big\}.$$

Example 4.1. *Consider the T2PFS's for* $X = (2, 3, 4)$,

$$\tilde{P}_1 = \left(\frac{<0.9, 0.0>}{0.8}, \frac{<0.7, 0.5>}{0.3}\right)\Big/_2 \sqcup \left(\frac{<0.8, 0.1>}{0.6}, \frac{<0.4, 0.2>}{0.6}\right)\Big/_3 \\ \sqcup \left(\frac{<0.9, 0.1>}{0.4}, \frac{<0.1, 0.8>}{0.7}\right)\Big/_4,$$

$$\tilde{P}_2 = \left(\frac{<0.8, 0.1>}{0.7}, \frac{<0.5, 0.5>}{0.9}\right)\Big/_2 \sqcup \left(\frac{<0.9, 0.1>}{0.4}, \frac{<0.5, 0.3>}{0.6}\right)\Big/_3 \\ \sqcup \left(\frac{<0.8, 0.2>}{0.5}, \frac{<0.8, 0.2>}{0.5}\right)\Big/_4,$$

To ease the computations, let us split the primary membership and non-membership measures and express them as ${\tilde{P}_1}^m$, ${\tilde{P}_1}^n$, ${\tilde{P}_2}^m$ *and* ${\tilde{P}_2}^n$:

$${\tilde{P}_1}^m = \left(\frac{<0.9, 0.0>}{0.8}\right)\Big/_2 \sqcup \left(\frac{<0.8, 0.1>}{0.6}\right)\Big/_3 \sqcup \left(\frac{<0.9, 0.1>}{0.4}\right)\Big/_4,$$

$${\tilde{P}_1}^n = \left(\frac{<0.7, 0.5>}{0.3}\right)\Big/_2 \sqcup \left(\frac{<0.4, 0.2>}{0.6}\right)\Big/_3 \sqcup \left(\frac{<0.1, 0.8>}{0.7}\right)\Big/_4,$$

$${\tilde{P}_2}^m = \left(\frac{<0.8, 0.1>}{0.7}\right)\Big/_2 \sqcup \left(\frac{<0.9, 0.1>}{0.4}\right)\Big/_3 \sqcup \left(\frac{<0.8, 0.2>}{0.5}\right)\Big/_4,$$

and

$${\tilde{P}_2}^n = \left(\frac{<0.5, 0.5>}{0.9}\right)\Big/_2 \sqcup \left(\frac{<0.5, 0.3>}{0.6}\right)\Big/_3 \sqcup \left(\frac{<0.8, 0.2>}{0.5}\right)\Big/_4.$$

Let $\vee$ *and* $\wedge$ *denote the fuzzy union (computed using the maximum operator) and fuzzy intersection (computed by the minimum operator) respectively, then,*

$$P^m = (\tilde{P}_1 \sqcup \tilde{P}_2)^m \\ = \left(\frac{<0.9 \vee 0.8, 0.0 \wedge 0.1>}{0.8 \vee 0.7}\right)\Big/_2 \sqcup \left(\frac{<0.8 \vee 0.9, 0.1 \wedge 0.1>}{0.6 \vee 0.4}\right)\Big/_3 \sqcup \left(\frac{<0.9 \vee 0.8, 0.1 \wedge 0.2>}{0.4 \vee 0.5}\right)\Big/_4 \\ = \left(\frac{<0.9, 0.0>}{0.8}\right)\Big/_2 \sqcup \left(\frac{<0.9, 0.1>}{0.6}\right)\Big/_3 \sqcup \left(\frac{<0.9, 0.1>}{0.5}\right)\Big/_4,$$

Similarly,

$$\begin{aligned} P^n &= (\tilde{P}_1 \sqcup \tilde{P}_2)^n \\ &= \left(\frac{<0.7 \vee 0.5, 0.5 \wedge 0.5>}{0.3 \wedge 0.9}\right)/_2 \sqcup \left(\frac{<0.4 \vee 0.5, 0.2 \wedge 0.3>}{0.6 \wedge 0.6}\right)/_3 \sqcup \left(\frac{<0.1 \vee 0.8, 0.8 \wedge 0.2>}{0.7 \wedge 0.5}\right)/_4 \\ &= \left(\frac{<0.7, 0.5>}{0.3}\right)/_2 \sqcup \left(\frac{<0.5, 0.2>}{0.6}\right)/_3 \sqcup \left(\frac{<0.8, 0.2>}{0.5}\right)/_4 \end{aligned}$$

Therefore,

$$\begin{aligned} \tilde{P} &= \tilde{P}_1 \sqcup \tilde{P}_2 \\ &= \left(\frac{<0.9, 0.0>}{0.8}, \frac{<0.7, 0.5>}{0.3}\right)/_2 \sqcup \left(\frac{<0.9, 0.1>}{0.6}, \frac{<0.5, 0.2>}{0.6}\right)/_3 \\ &\quad \sqcup \left(\frac{<0.9, 0.1>}{0.5}, \frac{<0.8, 0.2>}{0.5}\right)/_4, \\ \tilde{P}_2 &= \tilde{P}_1 \bigcap \tilde{P}_2 \\ &= \left(\frac{<0.8, 0.1>}{0.7}, \frac{<0.5, 0.5>}{0.9}\right)/_2 \sqcup \left(\frac{<0.8, 0.1>}{0.4}, \frac{<0.4, 0.3>}{0.6}\right)/_3 \\ &\quad \sqcup \left(\frac{<0.8, 0.2>}{0.4}, \frac{<0.1, 0.8>}{0.7}\right)/_4, \\ \tilde{P_1}^c &= \left(\frac{<0.5, 0.7>}{0.3}, \frac{<0.0, 0.9>}{0.8}\right)/_2 \sqcup \left(\frac{<0.2, 0.4>}{0.6}, \frac{<0.1, 0.8>}{0.6}\right)/_3 \\ &\quad \sqcup \left(\frac{<0.8, 0.1>}{0.7}, \frac{<0.1, 0.9>}{0.4}\right)/_4. \end{aligned}$$

Theorem 4.1. *The properties below are valid for any T2PFS $\tilde{P}$, $\tilde{P}_1$, $\tilde{P}_2$ and $\tilde{P}_3$:*

(1) $\tilde{P}_1 \bigcup \tilde{P}_2 = \tilde{P}_2 \bigcup \tilde{P}_1$ *and* $\tilde{P}_1 \bigcap \tilde{P}_2 = \tilde{P}_2 \bigcap \tilde{P}_1$ (*commutativity*)

(2) $\left(\tilde{P}_1 \bigcup \tilde{P}_2\right) \bigcup \tilde{P}_3 = \tilde{P}_1 \bigcup \left(\tilde{P}_2 \bigcup \tilde{P}_3\right)$ *and* $\left(\tilde{P}_1 \bigcap \tilde{P}_2\right) \bigcap \tilde{P}_3 = \tilde{P}_1 \bigcap \left(\tilde{P}_2 \bigcap \tilde{P}_3\right)$ (*associativity*)

(3) $\left(\tilde{P}^c\right)^c = \tilde{P}$ (*involution*)

(4) $\left(\tilde{P}_1 \bigcup \tilde{P}_2\right)^c = \tilde{P_1}^c \bigcap \tilde{P_2}^c$ *and* $\left(\tilde{P}_1 \bigcap \tilde{P}_2\right)^c = \tilde{P_1}^c \bigcup \tilde{P_2}^c$ (*De Morgan's law*)

(5) $\tilde{P} \bigcup \tilde{P} = \tilde{P}$ (*idempotency*)

(6) $\tilde{P}_1 \bigcup \left(\tilde{P}_2 \bigcap \tilde{P}_3\right) = \left(\tilde{P}_1 \bigcup \tilde{P}_2\right) \bigcap \left(\tilde{P}_1 \bigcup \tilde{P}_3\right)$ *and* $\tilde{P}_1 \bigcap \left(\tilde{P}_2 \bigcup \tilde{P}_3\right) = \left(\tilde{P}_1 \bigcap \tilde{P}_2\right) \bigcup \left(\tilde{P}_1 \bigcap \tilde{P}_3\right)$ (*distributive law*)

Proof.

(1) Since the fuzzy union and intersection of the primary membership measures $u_1(y)$, $u_2(y)$, the primary non-membership measures $v_1(y)$, $v_2(y)$, the secondary membership measures $\alpha_{\tilde{P}_1}(u_1(y))$, $\alpha_{\tilde{P}_2}(u_2(y))$,$\alpha_{\tilde{P}_1}(v_1(y))$, $\alpha_{\tilde{P}_2}(v_2(y))$ and the secondary non-membership measures

$\beta_{\tilde{P}_1}(u_1(y))$, $\beta_{\tilde{P}_2}(u_2(y))$,$\beta_{\tilde{P}_1}(v_1(y))$, $\beta_{\tilde{P}_2}(v_2(y))$ are all commutative,

$$
\begin{aligned}
\tilde{P}_1 \bigcup \tilde{P}_2 = \Bigg(< &\bigvee_{\bigvee_{y\in X}(u_1(y),u_2(y))} \big(\alpha_{\tilde{P}_1}(u_1(y)), \alpha_{\tilde{P}_2}(u_2(y))\big), \\
&\bigwedge_{\bigvee_{y\in X}(u_1(y),u_2(y))} \big(\beta_{\tilde{P}_1}(u_1(y)), \beta_{\tilde{P}_2}(u_2(y))\big) > / _{\bigvee_{y\in X}(u_1(y),u_2(y))}\Bigg) \bigvee_{y\in X} \\
&\Bigg(< \bigvee_{\bigwedge_{y\in X}(v_1(y),v_2(y))} \big(\alpha_{\tilde{P}_1}(v_1(y)), \alpha_{\tilde{P}_2}(v_2(y))\big), \\
&\bigwedge_{\bigwedge_{y\in X}(v_1(y),v_2(y))} \big(\beta_{\tilde{P}_1}(v_1(y)), \beta_{\tilde{P}_2}(v_2(y))\big) > / _{\bigwedge_{y\in X}(v_1(y),v_2(y))}\Bigg) \\
= \Bigg(< &\bigvee_{\bigvee_{y\in X}(u_2(y),u_1(y))} \big(\alpha_{\tilde{P}_2}(u_2(y)), \alpha_{\tilde{P}_1}(u_1(y))\big), \\
&\bigwedge_{\bigvee_{y\in X}(u_2(y),u_1(y))} \big(\beta_{\tilde{P}_2}(u_2(y)), \beta_{\tilde{P}_1}(u_1(y))\big) > / _{\bigvee_{y\in X}(u_2(y),u_1(y))}\Bigg) \bigvee_{y\in X} \\
&\Bigg(< \bigvee_{\bigwedge_{y\in X}(v_2(y),v_1(y))} \big(\alpha_{\tilde{P}_2}(v_2(y)), \alpha_{\tilde{P}_1}(v_1(y))\big), \\
&\bigwedge_{\bigwedge_{y\in X}(v_2(y),v_1(y))} \big(\beta_{\tilde{P}_2}(v_2(y)), \beta_{\tilde{P}_1}(v_1(y))\big) > / _{\bigwedge_{y\in X}(v_2(y),v_1(y))}\Bigg) \\
= \tilde{P}_2 \bigcup \tilde{P}_1.
\end{aligned}
$$

Similarly, we can demonstrate the commutativity of the intersection of two T2PFS's.

(2) The proof of this case can be obtained by the extension of Case (1).

(3) We know that,

$$\tilde{P}^c = \Big\{ < \big(y, v(y), u(y)\big), \big(v(y), \beta_{\tilde{P}}(v(y)), \alpha_{\tilde{P}}(v(y))\big), \big(u(y), \beta_{\tilde{P}}(u(y)), \alpha_{\tilde{P}}(u(y))\big) > \Big\}.$$

Therefore,

$$(\tilde{P}^c)^c = \Big\{ < \big(y, u(y), v(y)\big), \big(u(y), \alpha_{\tilde{P}}(u(y)), \beta_{\tilde{P}}(u(y))\big), \big(v(y), \alpha_{\tilde{P}}(v(y)), \beta_{\tilde{P}}(v(y))\big) > \Big\} = \tilde{P}.$$

(4) Since the fuzzy union and intersection of the primary membership measures $u_1(y)$, $u_2(y)$, the primary non-membership measures $v_1(y)$, $v_2(y)$, the secondary membership measures $\alpha_{\tilde{P}_1}(u_1(y))$, $\alpha_{\tilde{P}_2}(u_2(y))$,$\alpha_{\tilde{P}_1}(v_1(y))$, $\alpha_{\tilde{P}_2}(v_2(y))$ and the secondary non-membership measures

$\beta_{\tilde{P}_1}(u_1(y))$, $\beta_{\tilde{P}_2}(u_2(y))$,$\beta_{\tilde{P}_1}(v_1(y))$, $\beta_{\tilde{P}_2}(v_2(y))$ satisfy De Morgan's law,

$$\begin{aligned}
\left(\tilde{P}_1 \bigcup \tilde{P}_2\right)^c = \Bigg[\Bigg(< &\bigvee_{\bigvee_{y\in X}(u_1(y),u_2(y))} \left(\alpha_{\tilde{P}_1}(u_1(y)), \alpha_{\tilde{P}_2}(u_2(y))\right), \\
&\bigwedge_{\bigvee_{y\in X}(u_1(y),u_2(y))} \left(\beta_{\tilde{P}_1}(u_1(y)), \beta_{\tilde{P}_2}(u_2(y))\right) > / _{\bigvee_{y\in X}(u_1(y),u_2(y))}\Bigg) \bigvee_{y\in X} \\
&\Bigg(< \bigvee_{\bigwedge_{y\in X}(v_1(y),v_2(y))} \left(\alpha_{\tilde{P}_1}(v_1(y)), \alpha_{\tilde{P}_2}(v_2(y))\right), \\
&\bigwedge_{\bigwedge_{y\in X}(v_1(y),v_2(y))} \left(\beta_{\tilde{P}_1}(v_1(y)), \beta_{\tilde{P}_2}(v_2(y))\right) > / _{\bigwedge_{y\in X}(v_1(y),v_2(y))}\Bigg)\Bigg]^c \\
= \Bigg[\Bigg(< &\bigwedge_{\bigwedge_{y\in X}(v_1(y),v_2(y))} \left(\beta_{\tilde{P}_1}(v_1(y)), \beta_{\tilde{P}_2}(v_2(y))\right), \\
&\bigvee_{\bigwedge_{y\in X}(v_1(y),v_2(y))} \left(\alpha_{\tilde{P}_1}(v_1(y)), \alpha_{\tilde{P}_2}(v_2(y))\right) > / _{\bigwedge_{y\in X}(v_1(y),v_2(y))}\Bigg) \bigwedge_{y\in X} \\
&\Bigg(< \bigwedge_{\bigvee_{y\in X}(u_1(y),u_2(y))} \left(\beta_{\tilde{P}_1}(u_1(y)), \beta_{\tilde{P}_2}(u_2(y))\right), \\
&\bigvee_{\bigvee_{y\in X}(u_1(y),u_2(y))} \left(\alpha_{\tilde{P}_1}(u_1(y)), \alpha_{\tilde{P}_2}(u_2(y))\right) > / _{\bigvee_{y\in y}(u_1(y),u_2(y))}\Bigg)\Bigg] \\
= \tilde{P}_1{}^c \bigcap \tilde{P}_2{}^c.
\end{aligned}$$

Similarly, $\left(\tilde{P}_1 \bigcap \tilde{P}_2\right)^c = \tilde{P}_1{}^c \bigcup \tilde{P}_2{}^c$.

The proof for the last two case is straight forward, as fuzzy union and fuzzy intersection adhere to both idempotency and distributive law.

□

Remark 4.1. *Properties 1 through 4 of Theorem 4.1 are fulfilled by both T-norms and T-conorms as they are inherently commutative, associative, involuntary and also conform to De Morgan's law by definition. However, it is important to note that the last two properties may not always hold true, as exemplified below in Example 4.2.*

Example 4.2. *Consider the T-conorm(drastic union) $C^*$and T-norm(drastic intersection) T^* [14] defined by:*

$$C^*(u,v) = \begin{cases} u, & \text{if } v = 0, \\ v, & \text{if } u = 0, \\ 1, & \text{otherwise.} \end{cases}$$

and

$$T^*(u,v)=\begin{cases}u, & \text{if } v=1,\\ v, & \text{if } u=1,\\ 0, & \text{otherwise}.\end{cases}$$

Then, we shall show that these do not satisfy properties 5 and 6 of Theorem 4.1.
Suppose $\tilde{P} =< (1.0, 0.2, 0.5)(0.2, 0.8, 0.3)(0.5, 0.9, 0.0) >$.
Then,

$$\begin{aligned}\tilde{P}\bigcup\tilde{P} =&< \big(1, C^*(0.2,0.2), T^*(0.5,0.5)\big), \big(C^*(0.2,0.2), C^*(0.8,0.8), T^*(0.3,0.3)\big),\\ &\big(T^*(0.5,0.5), C^*(0.9,0.9), T^*(0,0)\big) >\\ =&< (1,1,0)(1,1,0)(0,1,0) >\neq \tilde{P}.\end{aligned}$$

which shows that these operators do not satisfy idempotency.
Now suppose,

$$\begin{aligned}\tilde{P}_1 &=< (1.0, 0.2, 0.5)(0.2, 0.8, 0.3)(0.5, 0.9, 0.0) >,\\ \tilde{P}_2 &=< (1.0, 0.3, 0.1)(0.3, 0.0, 1.0)(0.1, 0.8, 0.1) >,\\ \tilde{P}_3 &=< (1.0, 0.5, 0.6)(0.5, 1.0, 0.0)(0.6, 0.0, 0.9) > .\end{aligned}$$

Then,

$$\begin{aligned}\tilde{P}_2\bigcap\tilde{P}_3 =&< \big(1, T^*(0.3,0.5), C^*(0.1,0.6)\big), \big(T^*(0.3,0.5), T^*(0.0,1.0), C^*(1.0,0.0)\big),\\ &\big(C^*(0.1,0.6), T^*(0.8,0.0), C^*(0.1,0.9)\big) >\\ =&< (1,0,1)(0,0,1)(1,0,1) > .\end{aligned}$$

Similarly,

$$\begin{aligned}\tilde{P}_1\bigcup\Big(\tilde{P}_2\bigcap\tilde{P}_3\Big) =&< \big(1, C^*(0.2,0.0), T^*(0.5,1.00)\big), \big(C^(0.2,0.0), C^*(0.8,0.0), T^*(0.3,1.0)\big),\\ &\big(T^*(0.5,1.0), C^*(0.9,0.0), T^*(0.0,1.0)\big) >\\ =&< (1.0, 0.2, 0.5)(0.2, 0.8, 0.3)(0.5, 0.9, 0.0) >\end{aligned}$$

Now computing in a similar manner,

$$\begin{aligned}\tilde{P}_1\bigcup\tilde{P}_2 &=< (1.0, 1.0, 0.0)(1.0, 0.8, 0.3)(0.0, 1.0, 0.0) > \qquad \text{and},\\ \tilde{P}_1\bigcup\tilde{P}_3 &=< (1.0, 1.0, 0.0)(1.0, 1.0, 0.0)(0.0, 1.0, 0.0) >\end{aligned}$$

Therefore,

$$\Big(\tilde{P}_1\bigcup\tilde{P}_2\Big)\bigcap\Big(\tilde{P}_1\bigcup\tilde{P}_3\Big) =< (1.0, 1.0, 0.0)(1.0, 0.8, 0.3)(0.0, 1.0, 0.0) >\neq \tilde{P}_1\bigcup\Big(\tilde{P}_2\bigcap\tilde{P}_3\Big)$$

Certainly, it's evident that the second distributive law is not met.
Likewise, we can establish that the remaining T-conorms and T-norms also fail to exhibit idempotency and comply with distributive laws.

Definition 4.4 (Absolute T2PFS and Null T2PFS)**.** *An absolute T2PFS is defined as,*

$$\tilde{P}_A = \{< (y,1,0),(1,1,0),(0,1,0) >: \forall y \in X\},$$

and a Null T2PFS is defined as,

$$\tilde{P}_\emptyset = \{< (y,0,1),(0,0,1),(1,0,1) >: \forall y \in X\}.$$

Theorem 4.2.

$$\tilde{P}_A^c = \tilde{P}_\emptyset.$$

Definition 4.5 (Score Funtion)**.** *Let a T2PFN be given as,*

$$\tilde{p} =< \Big(y, u(y), v(y)\Big), \Big(u(y), \alpha(u(y)), \beta(u(y))\Big), \Big(v(y), \alpha(v(y)), \beta(v(y))\Big) >,$$

then the score function of $\tilde{p}$ is defined as,

$$Sco(\tilde{p}) = \frac{u^2(y) - v^2(y) + u^2(y)|\alpha^2(u(y)) - \beta^2(u(y))| - v^2(y)|\alpha^2(v(y)) - \beta^2(v(y))|}{2}.$$

Definition 4.6 (Hesitancy)**.** *Let a T2PFN be given as,*

$$\tilde{p} =< \Big(y, u(y), v(y)\Big), \Big(u(y), \alpha(u(y)), \beta(u(y))\Big), \Big(v(y), \alpha(v(y)), \beta(v(y))\Big) >,$$

then the Hesitancy of $\tilde{p}$ is defined as,

$$Hes(\tilde{p}) = 1 - u^2(y)\Big(\alpha^2(u(y)) + \beta^2(u(y))\Big) - v^2(y)\Big(\alpha^2(v(y)) + \beta^2(v(y))\Big).$$

Theorem 4.3. *Suppose $\tilde{p}$ is any T2PFN of a T2PFS $\tilde{P}$ and $\tilde{p}_A$ and $\tilde{p}_\emptyset$ are any T2PFNs of absolute T2PFS $\tilde{P}_A$ and null T2PFS $\tilde{P}_\emptyset$, respectively, defined in Definition 4.4. Then,*

(1) $-1 \leq Sco(\tilde{p}) \leq 1 \;\forall\, \tilde{p} \in \tilde{P}$.

(2) $Sco(\tilde{p}_A) = 1 \;\forall\, \tilde{p}_A \in \tilde{P}_A$ *and* $Sco(\tilde{p}_\emptyset) = -1 \;\forall\, \tilde{p}_\emptyset \in \tilde{P}_\emptyset$.

(3) $Sco(\tilde{p}^c) = -Sco(\tilde{p}) \;\forall\, \tilde{p} \in \tilde{P}$.

(4) $0 \leq Hes(\tilde{p}) \leq 1 \;\forall\, \tilde{p} \in \tilde{P} \;\forall\, \tilde{p} \in \tilde{P}$.

(5) $Hes(\tilde{p}_A) = 0 \;\forall\, \tilde{p}_A \in \tilde{P}_A$ *and* $Hes(\tilde{p}_\emptyset) = 0 \;\forall\, \tilde{p}_\emptyset \in \tilde{P}_\emptyset$.

(6) $Hes(\tilde{p}^c) = Hes(\tilde{p}) \;\forall\, \tilde{p} \in \tilde{P}$.

Proof.

(1)

$$0 \leq u^2(y) \leq 1,$$
$$0 \leq |\alpha^2_{\tilde{P}}(u(y)) - \beta^2_{\tilde{P}}(u(y))| \leq 1.$$

Therefore,

$$0 \leq u^2(y)|\alpha^2_{\tilde{P}}(u(y)) - \beta^2_{\tilde{P}}(u(y))| \leq 1.$$

Similarly,

$$0 \leq v^2(y) \leq 1,$$

$$0 \leq |\alpha^2_{\tilde{P}}(v(y)) - \beta^2_{\tilde{P}}(v(y))| \leq 1.$$

Therefore,

$$0 \leq v^2(y)|\alpha^2_{\tilde{P}}(v(y)) - \beta^2_{\tilde{P}}(v(y))|\} \leq 1$$

Also,

$$-1 \leq u^2(y) - v^2(y) \leq 1$$

Maximum of

$$\{u^2(y) - v^2(y)\} + \{u^2(y)|\alpha^2_{\tilde{P}}(u(y)) - \beta^2_{\tilde{P}}(u(y))|\} - \{v^2(y)|\alpha^2_{\tilde{P}}(v(y)) - \beta^2_{\tilde{P}}(v(y))|\}$$

is 2 and minimum of the same is -2.
Which means, maximum of

$$\frac{\{u^2(y) - v^2(y)\} + \left\{u^2(y)|\alpha^2_{\tilde{P}}(u(y)) - \beta^2_{\tilde{P}}(u(y))|\right\} - \left\{v^2(y)|\alpha^2_{\tilde{P}}(v(y)) - \beta^2_{\tilde{P}}(v(y))|\right\}}{2}$$

is 1 and minimum of the same is -1.
Hence, by Definition 5.1,

$$-1 \leq Sco(\tilde{p}) \leq 1, \forall\, \tilde{p} \in \tilde{P}.$$

(2) Consider $\tilde{P}_A$ and $\tilde{P}_\emptyset$ as in Definition 4.4.
For $\tilde{p}_A \in \tilde{P}_A$,

$$Sco(\tilde{p}_A) = \frac{1 - 0 + 1|1 - 0| - 0|1 - 0|}{2} = 1,$$

and for $\tilde{p}_\emptyset \in \tilde{P}_\emptyset$,

$$Sco(\tilde{p}_\emptyset) = \frac{0 - 1 + 0|0 - 1| - 1|0 - 1|}{2} = -1.$$

(3) We know, by Definition 4.5,

$$Sco(\tilde{p}) = \frac{u^2(y) - v^2(y) + u^2(y)|\alpha^2(u(y)) - \beta^2(u(y))| - v^2(y)|\alpha^2(v(y)) - \beta^2(v(y))|}{2}.$$

Hence, by applying Definition 4.3 of complement of a T2PFN,

$$\begin{aligned} Sco(\tilde{p}^c) &= \frac{v^2(y) - u^2(y) + v^2(y)|\beta^2(v(y)) - \alpha^2(v(y))| - u^2(y)|\beta^2(v(y)) - \alpha^2(v(y))|}{2}, \\ &= -\frac{u^2(y) - v^2(y) - v^2(y)|\alpha^2(v(y)) - \beta^2(v(y))| + u^2(y)|\alpha^2(v(y)) - \beta^2(v(y))|}{2} \\ &= -Sco(\tilde{p}). \end{aligned}$$

(4) We have,

$$0 \leq \alpha^2_{\tilde{P}}(u(y)) + \beta^2_{\tilde{P}}(u(y)) \leq 1.$$

Therefore,

$$0 \leq u^2(y)(\alpha^2_{\tilde{P}}(u(y)) - \beta^2_{\tilde{P}}(u(y))) \leq u^2(y).$$

Also,

$$0 \leq \alpha^2_{\tilde{P}}(v(y)) + \beta^2_{\tilde{P}}(v(y)) \leq 1.$$

Therefore,

$$0 \leq v^2(y)(\alpha^2_{\tilde{P}}(v(y)) - \beta^2_{\tilde{P}}(v(y))) \leq v^2(y).$$

Adding the above, we get,

$$0 \leq u^2(y)(\alpha^2_{\tilde{P}}(u(y)) - \beta^2_{\tilde{P}}(u(y))) + v^2(y)(\alpha^2_{\tilde{P}}(v(y)) - \beta^2_{\tilde{P}}(v(y))) \leq u^2(y) + v^2(y).$$

Hence, by Definition 4.6, $0 \leq H(\tilde{p}) \leq 1 \; \forall \, \tilde{p} \in \tilde{P}$.

(5) Consider $\tilde{P}_A$ and $\tilde{P}_\emptyset$ as in Definition 4.4.
For $\tilde{p}_A \in \tilde{P}_A$,

$$Hes(\tilde{p}_A) = 1 - 1(1+0) - 0(1-0) = 0,$$

and for $\tilde{p}_\emptyset \in \tilde{P}_\emptyset$,

$$Hes(\tilde{p}_\emptyset) = 1 - 0(0+1) - 1(0+1) = 0.$$

(6) We know, by Definition 4.6,

$$Hes(\tilde{p}) = 1 - u^2(y)(\alpha^2(u(y)) + \beta^2(u(y))) - v^2(y)(\alpha^2(v(y)) + \beta^2(v(y))).$$

Hence, by applying Definition 4.3 of complement of a T2PFN,

$$\begin{aligned} Hes(\tilde{p}^c) &= 1 - v^2(y)(\beta^2(v(y)) + \alpha^2(v(y))) - u^2(y)\left(\beta^2(u(y)) + \alpha^2(u(y))\right) \\ &= Hes(\tilde{p}). \end{aligned}$$

□

5 Application Examples Using T2PFN

In this section, we explore an algorithm designed to address real life decision-making problems. Initially, we present the evaluation function and then use a T2PFS ranking using the score function Definition 4.5 and hesitancy Definition 4.6 to effectively resolve the problem. We also illustrate the algorithm using an interview based scenario.

Suppose $X = \{y_1, y_2, \ldots, y_n\}$ is the set of alternatives and $C = \{C_1, C_2, \ldots, C_m\}$ is the set of criteria. Assume that the magnitude with which alternative y_j satisfies the criterion C_i is given as a T2PFN,

$$\tilde{p}_{ij} =< (y_j, u_{ij}, v_{ij}), (u_{ij}, {\alpha_{ij}}^u, {\beta_{ij}}^v), (v_{ij}, {\alpha_{ij}}^v, {\beta_{ij}}^v) >,$$

for $1 \leq i \leq m$ and $1 \leq j \leq n$, then the characteristics of the alternative y_j can be considered as the T2PFS,

$$\tilde{P}_j = \bigsqcup_{y_j \in X} \left(\bigsqcup_{u_{ij}, v_{ij}} \frac{< {\alpha_{ij}}^u, {\beta_{ij}}^u >}{u_{ij}}, \frac{< {\alpha_{ij}}^v, {\beta_{ij}}^v >}{v_{ij}} \right) \Big/ _{y_j}.$$

These criteria and the magnitude to which the alternatives satisfy each of them can be represented as T2PFNs as in Table1. It should be noted that the ij^{th} box of the table represents the T2PFN giving the magnitude with which alternative y_j satisfies the criterion C_i while j^{th} column of the table gives the T2PFS representing the characteristic of the alternative y_j.

Definition 5.1 (Evaluation Function)**.** [5] *When the decision maker requires the alternatives to satisfy the criteria $C_k, C_{k+1}, C_{k+2}, \ldots$ and C_l from the set of criteria C mentioned above, then we define an evaluation function of the j^{th} alternative as $\tilde{E}_j = \tilde{E}(\tilde{P}_j) = \bigcap_{i=k}^{l} \tilde{p_{ij}}$, where the $\tilde{p_{ij}}$'s are the T2PFNs representing the magnitudes to which the alternative $\tilde{y}_j$ satisfies the i^{th} criterion C_i. Suppose that the decision maker requires either of the following criteria, say $C_k, C_{k+1}, C_{k+2}, \ldots$ or C_l, the evaluation function is treated as the union of the required criteria, that is, the evaluation function is defined as $\tilde{E}_j = \bigcup_{i=k}^{l} \tilde{p_{ij}}$. If the decision maker requires the criteria $C_k, C_{k+1}, C_{k+2}, \ldots$ and C_l or C_s, then the evaluation function is defined as $\tilde{E}_j = \bigcap_{i=k}^{l} \tilde{p_{ij}} \bigcup \tilde{p_{sj}}$ and so forth.*

Definition 5.2. *Let $\tilde{E}_j$ be the evaluation function of the j^{th} alternative y_j. The ranking and preference of alternatives can be done using the score function and hesitancy as follows:*

1. *If $Sco(\tilde{E}_1) < Sco(\tilde{E}_2)$, then $\tilde{E}_1 \prec \tilde{E}_2$ meaning y_2 is preferred over y_1.*
2. *If $Sco(\tilde{E}_1) = Sco(\tilde{E}_2)$, then,*
   (a) *$Hes(\tilde{E}_1) > Hes(\tilde{E}_2)$ implies $\tilde{E}_1 \prec \tilde{E}_2$ meaning y_2 is preferred over y_1.*
   (b) *$Hes(\tilde{E}_1) = Hes(\tilde{E}_2)$ implies $\tilde{E}_1 \equiv \tilde{E}_2$ meaning y_1 and y_2 are equivalent.*

5.1 Algorithm

A ranking of alternatives can be found in a decision-making problem to determine the best alternative that meets the required criteria using the following algorithm. Suppose the situation has been analyzed and the required criteria have been identified. Then, the degree to which each alternative satisfies each criterion can be listed out, as shown in Table 1.

Step 1: Suppose there are criteria to be maximised and those to be minimised. Then the entries in the table need to be normalised as follows: Let $\tilde{p_{ij}}$ denote the T2PFN corresponding to the ith criterion and the jth alternative.Then,

$$\tilde{p_{ij}} = \begin{cases} \tilde{p_{ij}}, & \text{for maximising criterion,} \\ \tilde{p_{ij}}^{c}, & \text{for minimising criterion.} \end{cases} \tag{1}$$

Step 2: Using the above normalised $\tilde{p_{ij}}$'s find the evaluation function according to the choice of the criteria as in Definition 5.1.

Step 3: Find the score function of the evaluation function corresponding to each alternative using Definition 4.5.

Step 4: The alternative having maximum score for the evaluation function is the desired alternative. If there are alternatives which possess similar maximum score, then determine the hesitancy of the evaluation function of the alternatives and then compare their rankings as in Definition 5.2 to identify the optimal alternative.

Example 5.1. *Consider a scenario where an interview panel assesses 5 candidates say y_1, y_2, y_3, y_4 and y_5. The panel evaluates these candidates based on 6 key criteria: C_1 being the technical competencies (hard skills) required for the job, C_2 being the professional experience, C_3 the educational qualifications and relevant certifications, C_4 the soft skills, C_5 the interview performance and C_6 the performance of the candidate in group discussion setting. Then, $X = \{y_1, y_2, y_3, y_4, y_5\}$ is the set of alternatives and $C = \{C_1, C_2, C_3, C_4, C_5, C_6\}$ is the set of criteria. Suppose the magnitude with which the alternatives satisfy the criteria are as shown in the following table,*

Table 1: Attribute values of the alternatives under various criteria.

	$\tilde{P}_1$	$\tilde{P}_2$	$\tilde{P}_3$	$\tilde{P}_4$	$\tilde{P}_5$
C_1	$<(y_1,0.5,0.2), (0.5,1,0), (0.2,0,1)>$	$<(y_2,0.6,0.7), (0.6,0.7,0.2), (0.7,0.6,0.5)>$	$<(y_3,0.4,0.5), (0.4,0.5,0.8), (0.4,0.9,0.1)>$	$<(y_4,0.3,0.4), (0.3,0.7,0.2), (0.4,0.3,0.5)>$	$<(y_5,0.2,0.5), (0.2,0.3,0.7), (0.5,0.2,0.9)>$
C_2	$<(y_1,0.8,0.3), (0.8,0.5,0.5), (0.3,0.7,0.2)>$	$<(y_2,0.9,0.3), (0.9,0.2,0.5), (0.3,0.5,0.5)>$	$<(y_3,1,0), (1,1,0), (0,1,0)$	$<(y_4,0.9,0.1), (0.9,0.2,0.1), (0.1,0.2,0.1)>$	$<(y_5,1,0), (1,0.5,0.2), (0,0.7,0.2)>$
C_3	$<(y_1,0.5,0.1), (0.5,0.6,0), (0.1,0,1)>$	$<(y_2,1,0), (1,0.8,0.3), (0,0.6,0.3)>$	$<(y_3,0.6,0.1), (0.6,0.6,0.4), (0.1,0.9,0.1)$	$<(y_4,1,0), (1,1,0), (0,1,0)>$	$<(y_5,0.1,0.9), (0.1,0.2,0.7), (0.9,0.7,0.2)>$
C_4	$<(y_1,0.9,0.1), (0.9,0.8,0.3), (0.1,0.3,0.5)>$	$<(y_2,0.8,0.2), (0.8,0.9,0.2), (0.2,0.7,0.2)>$	$<(y_3,0.7,0.1), (0.7,0.7,0.3), (0.1,1,0)$	$<(y_4,0.9,0.1), (0.9,0.5,0.2), (0.1,0.3,0.6)>$	$<(y_5,0,1), (0,0,1), (1,0,1)>$
C_5	$<(y_1,0.6,0.2), (0.6,0.7,0.1), (0.2,0.2,0.7)>$	$<(y_2,0.7,0.4), (0.7,1,0), (0.4,0.8,0.1)>$	$<(y_3,0.8,0.2), (0.8,0.6,0.2), (0.2,0.9,0.1)$	$<(y_4,0.6,0.5), (0.6,0.3,0.7), (0.5,0.8,0.1)>$	$<(y_5,0.5,0.5), (0.5,0.4,0.4), (0.5,0.3,0.3)>$
C_6	$<(y_1,0.7,0.3), (0.7,0.7,0.4), (0.3,0.4,0.5)>$	$<(y_2,0.9,0.4), (0.9,0.6,0.4), (0.4,0.9,0.1)>$	$<(y_3,0.7,0.2), (0.7,0.8,0.1), (0.2,0.95,0.01)$	$<(y_4,0.4,0.2), (0.4,0.3,0.5), (0.2,0.2,0.7)>$	$<(y_5,0.75,0.25), (0.75,0.2,0.3), (0.25,0.8,0.3)>$

Assume the decision makers desire the fulfillment of all criteria. Then, the Algorithm 5.1 is carried out in the steps detailed below:

Step 1: *Since the criteria are to be maximised, the $\tilde{p_{ij}}$s remain the same.*

Step 2: *The evaluation functions are found as the intersection of values of all criteria.*

$$\tilde{E}_1 = \tilde{p_{11}} \bigcap \tilde{p_{21}} \bigcap \tilde{p_{31}} \bigcap \tilde{p_{41}} \bigcap \tilde{p_{51}} \bigcap \tilde{p_{61}} =< (y_1, 0.5, 0.3), (0.5, 0.5, 0.5), (0.3, 0.0, 1.0) >,$$

$$\tilde{E}_2 = \tilde{p_{12}} \bigcap \tilde{p_{22}} \bigcap \tilde{p_{32}} \bigcap \tilde{p_{42}} \bigcap \tilde{p_{52}} \bigcap \tilde{p_{62}} =< (y_2, 0.6, 0.7), (0.6, 0.2, 0.5), (0.7, 0.5, 0.5) >,$$

$$\tilde{E}_3 = \tilde{p_{13}} \bigcap \tilde{p_{23}} \bigcap \tilde{p_{33}} \bigcap \tilde{p_{43}} \bigcap \tilde{p_{53}} \bigcap \tilde{p_{63}} =< (y_3, 0.4, 0.5), (0.4, 0.5, 0.8), (0.5, 0.9, 0.1) >,$$

$$\tilde{E}_4 = \tilde{p_{14}} \bigcap \tilde{p_{24}} \bigcap \tilde{p_{34}} \bigcap \tilde{p_{44}} \bigcap \tilde{p_{54}} \bigcap \tilde{p_{64}} =< (y_4, 0.3, 0.5), (0.3, 0.2, 0.7), (0.5, 0.2, 0.7) >,$$

$$\tilde{E}_5 = \tilde{p_{15}} \bigcap \tilde{p_{25}} \bigcap \tilde{p_{35}} \bigcap \tilde{p_{45}} \bigcap \tilde{p_{55}} \bigcap \tilde{p_{65}} =< (y_5, 0.0, 1.0), (0.0, 0.0, 1.0), (1.0, 0.0, 1.0) > .$$

Step 3: *The score functions of each $\tilde{E}_i$'s are obtained as:*

$$Sco\{\tilde{E}_1\} = \frac{(0.5)^2 - (0.3)^2 + (0.5)^2|(0.5)^2 - (0.5)^2| - (0.3)^2|0.0 - 1.0|}{2} = 0.035,$$
$$Sco\{\tilde{E}_2\} = \frac{(0.6)^2 - (0.7)^2 + (0.6)^2|(0.2)^2 - (0.5)^2| - (0.7)^2|(0.5)^2 - (0.5)^2|}{2} = -0.02721,$$
$$Sco\{\tilde{E}_3\} = \frac{(0.4)^2 - (0.5)^2 + (0.4)^2|(0.5)^2 - (0.8)^2| - (0.5)^2|(0.9)^2 - (0.1)^2|}{2} = -0.0778,$$
$$Sco\{\tilde{E}_4\} = \frac{(0.3)^2 - (0.5)^2 + (0.3)^2|(0.2)^2 - (0.7)^2| - (0.5)^2|(0.2)^2 - (0.7)^2|}{2} = -0.116,$$
$$Sco\{\tilde{E}_5\} = \frac{(0.0)^2 - (1.0)^2 + (0.0)^2|(0.0)^2 - (1.0)^2| - (1.0)^2|(0.0)^2 - (1.0)^2|}{2} = -1.0.$$

Step 4: *Here, $Sco(\tilde{E}_1) > Sco(\tilde{E}_2) > Sco(\tilde{E}_3) > Sco(\tilde{E}_4) > Sco(\tilde{E}_5)$. Hence as per the algorithm, the candidate y_1 is deemed the top choice, followed by y_2, then y_3 and afer that y_4 with y_5 being the least preferred option.*

6 Discussion

In this section, we proceed with an extensive evaluation of our proposed algorithm in contrast to established methods in the field which uses score functions and accuracy. The primary aim is to showcase the superior efficacy and performance of our algorithm in addressing the specific challenges of decision-making scenarios. The example presents a comparison of results obtained from the proposed algorithm with those from existing methods developed in [28, 12]. A further comparison is also made between the proposed algorithm and the algorithms in [11, 21]. Let a T2PFN be given as,

$$\tilde{p} = \{< (y, u(y), v(y)), (u(y), \alpha_{\tilde{P}}(u(y)), \beta_{\tilde{P}}(u(y))), (v(y), \alpha_{\tilde{P}}(v(y)), \beta_{\tilde{P}}(v(y))) >\}.$$

In [28], the definition of the score function is

$$S(\tilde{p}) = u^2(y) - v^2(y).$$

And the ranking is given as,

1. If $S(\tilde{p_1}) < S(\tilde{p_2})$, then $\tilde{p_1} \prec \tilde{p_2}$ meaning y_2 is preferred over y_1.
2. If $S(\tilde{p_1}) = S(\tilde{p_2})$, then, $\tilde{p_1} \equiv \tilde{p_2}$ meaning y_1 and y_2 are equivalent.

In [12], the definition of the score is the same as in [28] and an accuracy function is defined as,

$$H(\tilde{p}) = u^2(y) + v^2(y).$$

The ranking is given as,

1. If $S(\tilde{p_1}) < S(\tilde{p_2})$, then $\tilde{p_1} \prec \tilde{p_2}$ meaning y_2 is preferred over y_1.
2. If $S(\tilde{p_1}) = S(\tilde{p_2})$, then,
   (a) $H(\tilde{p_1}) < H(\tilde{p_2})$ implies $\tilde{p_1} \prec \tilde{p_2}$ meaning y_2 is preferred over y_1.
   (b) $H(\tilde{p_1}) = H(\tilde{p_2})$ implies $\tilde{p_1} \equiv \tilde{p_2}$ meaning y_1 and y_2 are equivalent.

While in [11], in addition to the above defined score and accuracy, a novel score function was defined as,

$$M(\tilde{p}) = u^2(y) - \sqrt{1 - u^2(y) - v^2(y)}.$$

A new ranking was defined as,

1. If $S(\tilde{p_1}) < S(\tilde{p_2})$, then $\tilde{p_1} \prec \tilde{p_2}$ meaning y_2 is preferred over y_1.
2. If $S(\tilde{p_1}) = S(\tilde{p_2})$, then,
   (a) $H(\tilde{p_1}) < H(\tilde{p_2})$ implies $\tilde{p_1} \prec \tilde{p_2}$ meaning y_2 is preferred over y_1.
   (b) $H(\tilde{p_1}) = H(\tilde{p_2})$, then,
      i. $M(\tilde{p_1}) < M(\tilde{p_2})$ implies $\tilde{p_1} \prec \tilde{p_2}$ meaning y_2 is preferred over y_1.
      ii. $M(\tilde{p_1}) = M(\tilde{p_2})$ implies $\tilde{p_1} \equiv \tilde{p_2}$ meaning y_1 and y_2 are equivalent.

While in [21], the formulation of the score function is

$$Sc(\tilde{p}) = u^2(y) - v^2(y) - \frac{1}{4}\left|\left\{\alpha^2_{\tilde{P}}(u(y)) - \beta^2_{\tilde{P}}(u(y))\right\} - \left\{\alpha^2_{\tilde{P}}(v(y)) - \beta^2_{\tilde{P}}(v(y))\right\}\right|,$$

and accuracy function is

$$Ac(\tilde{p}) = u^2(y) + v^2(y) - \frac{1}{4}\left|\left\{\alpha^2_{\tilde{P}}(u(y)) + \beta^2_{\tilde{P}}(u(y))\right\} - \left\{\alpha^2_{\tilde{P}}(v(y)) + \beta^2_{\tilde{P}}(v(y))\right\}\right|,$$

while the ranking was defined as,

1. If $Sc(\tilde{p_1}) < Sc(\tilde{p_2})$, then $\tilde{p_1} \prec \tilde{p_2}$ meaning y_2 is preferred over y_1.
2. If $Sc(\tilde{p_1}) = Sc(\tilde{p_2})$, then,
   (a) $Ac(\tilde{p_1}) < Ac(\tilde{p_2})$ implies $\tilde{p_1} \prec \tilde{p_2}$ meaning y_2 is preferred over y_1.
   (b) $Ac(\tilde{p_1}) = Ac(\tilde{p_2})$ implies $\tilde{p_1} \equiv \tilde{p_2}$ meaning y_1 and y_2 are equivalent.

The following Table 2 shows three alternatives and the magnitude with which they satisfy the characteristic C_1.

Table 2: Different alternatives satisfying a criterion.

	$\tilde{p}_1$	$\tilde{p}_2$	$\tilde{p}_3$
C_1	$<(y_1, 0.80, 0.60),$ $(0.80, 0.60, 0.40),$ $(0.60, 0.30, 0.50)>$	$<(y_2, 0.80, 0.60),$ $(0.80, 0.30, 0.50),$ $(0.60, 0.60, 0.40)>$	$<(y_3, 0.55, 0.15),$ $(0.55, 0.30, 0.20),$ $(0.15, 0.90, 0.10)>$

The comparison between the score functions of the above data developed with Definition 4.5 and the prioritization of the alternatives are noted in the following Table 3.

Table 3: Comparison table.

	$\tilde{p_1}$	$\tilde{p_2}$	$\tilde{p_3}$	Ranking
[28]	$S(\tilde{p}_1) = 0.2800$	$S(\tilde{p}_2) = 0.2800$	$S(\tilde{p}_3) = 0.2800$	$\tilde{p_1} \equiv \tilde{p_2} \equiv \tilde{p_3}$
[12]	$S(\tilde{p}_1) = 0.2800$ $H(\tilde{p}_1) = 1.0000$	$S(\tilde{p}_2) = 0.2800$ $H(\tilde{p}_2) = 1.0000$	$S(\tilde{p}_3) = 0.2800$ $H(\tilde{p}_3) = 0.3250$	$\tilde{p_1} \equiv \tilde{p_2} \succ \tilde{p_3}$
[11]	$S(\tilde{p}_1) = 0.2800$ $H(\tilde{p}_1) = 1.0000$ $M(\tilde{p}_1) = 0.6400$	$S(\tilde{p}_2) = 0.2800$ $H(\tilde{p}_2) = 1.0000$ $M(\tilde{p}_2) = 0.6400$	$S(\tilde{p}_3) = 0.2800$ $H(\tilde{p}_3) = 0.3250$ $M(\tilde{p}_3) = 0.5190$	$\tilde{p_1} \equiv \tilde{p_2} \succ \tilde{p_3}$
[21]	$Sc(\tilde{p}_1) = 0.1900$ $Ac(\tilde{p}_1) = 0.9550$	$Sc(\tilde{p}_2) = 0.1900$ $Ac(\tilde{p}_2) = 0.9550$	$Sc(\tilde{p}_3) = 0.0925$ $Ac(\tilde{p}_3) = 0.1075$	$\tilde{p_1} \equiv \tilde{p_2} \succ \tilde{p_3}$
Using Algorithm 5.1	$Sco(\tilde{p}_1) = 0.1752$	$Sco(\tilde{p}_2) = 0.1552$	$Sco(\tilde{p}_3) = 0.1385625$	$\tilde{p_1} \succ \tilde{p_2} \succ \tilde{p_3}$

From Table 3, it is evident that the alternatives y_1, y_2 and y_3 are assigned the same rank using the score function proposed in [28]. In contrast, the method in [12], which utilizes both score and accuracy functions, ranks y_1 and y_2 as the top alternatives. The approach in [11] introduces a novel accuracy function but yields the same preferences as in [12], offering no further advancement. Likewise, the recent method in [21], despite proposing new score and accuracy functions for T2PFNs, fails to produce a unique preferred alternative. In comparison, the algorithm presented in this paper identifies y_1 as the most preferred alternative, thus offering a clearer and more decisive ranking.

6.1 Advantages of the proposed algorithm

While the approach in [21] introduces novel score and accuracy functions for T2PFNs, it yield identical results for distinct alternatives when just the secondary membership and non-membership values are interchanged-highlighting a potential limitation in discriminative ability. In contrast, the score and hesitancy functions proposed in this work are structurally more robust, making it highly unlikely-even for two distinct T2PFNs to yield the same score value, and even more so for both the score and hesitancy values to coincide. This significantly enhances the precision and reliability of the ranking outcomes, offering a more effective and nuanced algorithm for decision-making under uncertainty.

7 Conclusion

This article presents a structured and effective approach to decision-making using T2PFS, demonstrating superior capability in handling uncertainty compared to traditional fuzzy methods. The approach is proven to handle uncertainty more effectively compared to traditional methods, owing to the incorporation of six key components (the primary membership and non-membership, and the secondary membership and non-membership of both the primary measures). The comparative analysis validates the discriminative strength and practical applicability of the proposed algorithm. However, a limitation of this specific approach is that it assigns equal weight to all criteria, which may not align with scenarios where different criteria could have varying degrees of importance. To address this limitation, we plan to develop an extended algorithm that considers decision-making problems with weighted criteria. We shall also extend our study to finding correlation coefficients and distance measures, thereby constructing an algorithm to solve decision-making problems in pattern recognition as well as disease diagnosis using T2PFS. Our research agenda will include extending the proposed algorithm to incorporate PCFS and FPPFHsES as well. Additionally, integrating the proposed model with established MCDM techniques such as VIKOR, WSM, WPM, and TOPSIS will be explored in future work to broaden comparative analysis and practical applicability.

Acknowledgement The authors are grateful to anonymous referees for their remarks and suggestions that helped improve the manuscript.

Conflicts of Interest The authors declare no conflict of interest.

References

[1] E. H. A. Al-Sabri, M. Rahim, F. Amin, R. Ismail, S. Khan, A. M. Alanzi & H. A. E.-W. Khalifa (2023). Multi-criteria decision-making based on Pythagorean cubic fuzzy Einstein aggregation operators for investment management. *AIMS Mathematics*, *8*(7), 16961–16988. https://doi.org/10.3934/math.2023866.

[2] K. T. Atanassov. Intuitionistic fuzzy sets. VII ITKR Session, Sofia, June 1983. Deposed in Central Science and Technical Library of the Bulgarian Academy of Sciences, Report No. 1697/84 1983.

[3] K. T. Atanassov (1999). *Intuitionistic Fuzzy Sets: Theory and Applications*, volume 35 of *Studies in Fuzziness and Soft Computing*, chapter Interval Valued Intuitionistic Fuzzy Sets, pp. 139–177. Physica-Verlag HD, Heidelberg. https://doi.org/10.1007/978-3-7908-1870-3_2.

[4] K. T. Attanassov (1986). Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, *20*(1), 87–96. https://doi.org/10.1016/S0165-0114(86)80034-3.

[5] S. M. Chen & J. M. Tan (1994). Handling multicriteria fuzzy decision-making problems based on vague set theory. *Fuzzy Sets and Systems*, *67*(2), 163–172. https://doi.org/10.1016/0165-0114(94)90084-1.

[6] B. C. Cuòng, T. H. Anh & B. D. Hài (2012). Some operations on type-2 intuitionistic fuzzy sets. *Journal of Computer Science and Cybernetics*, *28*(3), 274–283. https://doi.org/10.15625/1813-9663/28/3/2607.

[7] S. Dan, M. B. Kar, S. Majumder, B. Roy, S. Kar & D. Pamucar (2019). Intuitionistic type-2 fuzzy set and its properties. *Symmetry*, *11*(6), Article ID: 808. https://doi.org/10.3390/sym11060808.

[8] P. A. Ejegwa, Y. Feng, S. Tang, J. M. Agbetayo & X. Dai (2023). New Pythagorean fuzzy-based distance operators and their applications in pattern classification and disease diagnostic analysis. *Neural Computing and Applications*, *35*(14), 10083–10095. https://doi.org/10.1007/s00521-022-07679-3.

[9] P. A. Ejegwa, A. Sarkar & I. C. Onyeke (2023). *New methods of computing correlation coefficient based on Pythagorean fuzzy information and their applications in disaster control and diagnostic analysis*, pp. 473–498. Springer International Publishing, Cham. https://doi.org/10.1007/978-3-031-35668-1_21.

[10] P. A. Ejegwa, S. Wen, Y. Feng, W. Zhang & J. Chen (2021). Some new Pythagorean fuzzy correlation techniques via statistical viewpoint with applications to decision-making problems. *Journal of Intelligent & Fuzzy Systems*, *40*(5), 9873–9886. https://doi.org/10.3233/JIFS-202469.

[11] H. Garg (2016). A novel accuracy function under interval-valued Pythagorean fuzzy environment for solving multicriteria decision making problem. *Journal of Intelligent & Fuzzy Systems*, *31*(1), 529–540. https://doi.org/10.3233/IFS-162165.

[12] H. Garg (2017). A new improved score function of an interval-valued Pythagorean fuzzy set based TOPSIS method. *International Journal for Uncertainty Quantification*, *7*(5), 463–474. https://doi.org/10.1615/Int.J.UncertaintyQuantification.2017020197.

[13] M. Ihsan, M. Saeed, A. Alburaikan & H. A. E.-W. Khalifa (2022). Product evaluation through multi-criteria decision making based on fuzzy parameterized Pythagorean fuzzy hypersoft expert set. *AIMS Mathematics*, *7*(6), 11024–11052. https://doi.org/10.3934/math.2022616.

[14] G. Klir & B. Yuan (1995). *Fuzzy Sets and Fuzzy Logic* volume 4. Prentice Hall, New Jersey.

[15] R. Li, P. A. Ejegwa, K. Li, I. Agaji, Y. Feng & I. C. Onyeke (2024). A new similarity function for Pythagorean fuzzy sets with application in football analysis. *AIMS Mathematics*, *9*(2), 4990–5014. https://doi.org/10.3934/math.2024242.

[16] J. M. Mendel & R. I. B. John (2002). Type-2 fuzzy sets made simple. *IEEE Transactions on Fuzzy Systems*, *10*(2), 117–127. https://doi.org/10.1109/91.995115.

[17] J. M. Mendel, R. I. John & F. Liu (2006). Interval type-2 fuzzy logic systems made simple. *IEEE Transactions on Fuzzy Systems*, *14*(6), 808–821. https://doi.org/10.1109/TFUZZ.2006.879986.

[18] A. Mondal & S. K. Roy (2022). Application of Choquet integral in interval type-2 Pythagorean fuzzy sustainable supply chain management under risk. *International Journal of Intelligent Systems*, *37*(1), 217–263. https://doi.org/10.1002/int.22623.

[19] X. Peng & Y. Yang (2016). Fundamental properties of interval-valued Pythagorean fuzzy aggregation operators. *International Journal of Intelligent Systems*, *31*(5), 444–487. https://doi.org/10.1002/int.21790.

[20] M. S. Saidin, L. S. Lee, M. R. A. Bakar & M. Z. Ahmad (2022). A new divergence measure based on fuzzy TOPSIS for solving staff performance appraisal. *Malaysian Journal of Mathematical Sciences*, *16*(3), 637–658. https://doi.org/10.47836/mjms.16.3.14.

[21] B. Sarkar, D. Chakraborty & A. Biswas (2023). Development of type-2 Pythagorean fuzzy set with its application to sustainable transport system selection. *Applied Soft Computing*, *142*, Article ID: 110332. https://doi.org/10.1016/j.asoc.2023.110332.

[22] A. M. Terna, P. A. Ejegwa & Y. Feng (2024). An enhanced Pythagorean fuzzy distance method with application in the selection of renewable energy source via MCDA. In *2024 International Conference on New Trends in Computational Intelligence (NTCI), 18–20 October 2024, Qingdao, China*, pp. 392–401. IEEE, New York. https://doi.org/10.1109/NTCI64025.2024.10776528.

[23] W. R. Wan Mohd, L. Abdullah, B. Yusoff, C. M. I. C. Taib & J. M. Merigo (2019). An integrated MCDM model based on Pythagorean fuzzy sets for green supplier development program. *Malaysian Journal of Mathematical Sciences*, *13*(S), 23–37.

[24] K. Wu, P. A. Ejegwa, Y. Feng, I. C. Onyeke, S. E. Johnny & S. Ahemen (2022). Some enhanced distance measuring approaches based on Pythagorean fuzzy information with applications in decision making. *Symmetry*, *14*(12), Article ID: 2669. https://doi.org/10.3390/sym14122669.

[25] R. R. Yager (2013). Pythagorean membership grades in multicriteria decision making. *IEEE Transactions on Fuzzy Systems*, *22*(4), 958–965. https://doi.org/10.1109/TFUZZ.2013.2278989.

[26] D. Yan, K. Wu, P. A. Ejegwa, X. Xie & Y. Feng (2023). Pythagorean fuzzy partial correlation measure and its application. *Symmetry*, *15*(1), Article ID: 216. https://doi.org/10.3390/sym15010216.

[27] L. A. Zadeh (1965). Fuzzy sets. *Information and Control*, *8*(3), 338–353. https://doi.org/10.1016/S0019-9958(65)90241-X.

[28] X. Zhang & Z. Xu (2014). Extension of TOPSIS to multiple criteria decision making with Pythagorean fuzzy sets. *International Journal of Intelligent Systems*, *29*(12), 1061–1078. https://doi.org/10.1002/int.21676.